Title: Exponents and Exponential Functions

Brief Overview:

The focus of this unit is on recognizing, applying, representing, and analyzing exponential forms. The unit will begin with an investigation of the rapid growth of an exponential function. The unit proceeds with the modeling of various exponential functions. Patterns of change using exponential functions will be compared with linear and quadratic functions. Students will create symbolic, graphic, and numerical models to extend their knowledge of exponents. Later properties of exponents will be developed to enable students to simplify expressions containing exponents. The graphing calculator is an essential tool for this unit.

NCTM 2000 Principles for School Mathematics:

- **Equity:** *Excellence in mathematics education requires equity - high expectations and strong support for all students.*
- **Curriculum:** *A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades.*
- **Teaching:** *Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well.*
- **Learning:** *Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge.*
- **Assessment:** *Assessment should support the learning of important mathematics and furnish useful information to both teachers and students.*
- **Technology:** *Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.*

Links to NCTM 2000 Standards:

- **Content Standards**

<u>Number and Operations</u>
The students will judge the effects of such operations as multiplication, division, and computing powers and roots on magnitudes and quantities.

<u>Algebra</u>
Students will interpret representations of a function and investigate rate of change and intercepts. They will use symbolic algebra to represent mathematical relationships. They also will be able to draw reasonable conclusions about the situation being modeled.

Data Analysis and Probability
Students will understand scatter plots, and use them to display data. They will determine regression equations that model the data using technological tools.

- **Process Standards**

Mathematics as Problem Solving, Reasoning and Proof, Communication, Connections, and Representation
These five process standards are threads that integrate throughout the unit, although they may not be specifically addressed in the unit. They emphasize the need to help students develop the processes that are the major means for doing mathematics, thinking about mathematics, understanding mathematics and communicating mathematics.

Links to Maryland High School Mathematics Core Learning Units:

Functions and Algebra

- **1.1.1**
Students will recognize, describe, and extend patterns and functional relationships that are expressed numerically, algebraically, and geometrically

- **1.1.2.**
Students will represent patterns and functional relationships in a table, as a graph, and by mathematical expression.

- **1.2.1**
Students will determine the equation for a line, solve linear equations, and describe the solutions using numbers, symbols, and graphs.

- **1.2.4**
Students will describe how the graphical model of a non-linear function represents a given problem and will estimate the solution.

Data Analysis and Probability

- **3.1.1**
Students will conduct an investigation that uses statistical methods to analyze data and communicate results.

- **3.2.2**
Students will make predictions by finding and using a line of best fit and a curve of best fit.

Grade/Level:

Grades 9-12, Algebra I

Duration/Length:

Three 90 minute periods or five 56 minute periods

Prerequisite Knowledge:

Students should have working knowledge of the following skills:

- Basic mathematical operations
- Constructing graphs, tables, and scatter plots on the graphing calculator
- Basic knowledge of exponents, bases, and exponential operations
- Knowledge of solving, graphing, and interpreting of linear functions

Student Outcomes:

- Students will gain a graphic, numerical and symbolic understanding of exponential functions.
- Students will be able to identify, write and solve problems involving exponential functions.
- Students will be able to solve real-world problems using exponents.

Materials/Resources/Printed Materials:

- Graphing Calculator
- Worksheets and Assessments

Development/Procedures:

Introduction

Students will be given several motivational activities to incite interest and discover basics of exponents.

Day 1

The students will work on projects that will lay the groundwork for the unit by developing skills enabling them to identify an exponent, use exponents, and to evaluate exponential expressions.

Day 2

The student will gain further practice on using exponents, building exponential models (including exponential growth), and classifying functions as exponential, linear, or quadratic.

Day 3

The student will be able to identify exponential decay, identify characteristics of growth vs. decay and solve problems involving the same.

Assessment:

This unit will include an assessment consisting of selected response, student-produced response, and brief constructed response. The tasks will assess how well students understand vocabulary, concepts, skills, and problem solving abilities of exponents and exponential functions.

Extension/Follow Up:

Students will be asked to find examples of exponential functions in real-world applications. They will be encouraged to look in newspapers, magazines, and Internet websites.

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Warm Up
(Transparency)

Introductory Problem: Given 5 and 4, use one mathematical operation to get the highest number.

Discussion Questions.

1. What is the meaning of an exponent?

2. Does $5^4 = 4^5$? If they are not, which is greater?

3. Would 3^6 be greater than, less than, or equal to 6^3?

4. Given two different whole numbers would you get a greater answer, if you use the smaller number or the larger number in the base? Are there any exceptions? (Hint: Use questions 2 and 3.)

Warm Up
(Answer Key)

Introductory Problem: Given 5 and 4, use one mathematical operation to get the highest number.

Discussion Questions.

1. What is the meaning of an exponent?
   Exponents are repeated multiplications. An exponent tells how many times the base is used as a factor. For example, $6^3 = 6 * 6 * 6$.

2. Does $5^4 = 4^5$? If they are not equal, which is greater?
   No it is not the same. $5^4 = 625$ and $4^5 = 1024$. Therefore exchanging the base and exponent is generally not commutative.

3. Would 3^6 be greater than or less than 6^3?
   $3^6 > 6^3$ since $3^6 = 729$ and $6^3 = 216$.

4. Given two different whole numbers would you get a greater answer, if you use the smaller number or the larger number in the base? Are there any exceptions?
   You would get a greater answer if you put the smaller number in the base. The exceptions would be found if at least one of the numbers is 0 or 1. Also note $2^4 = 4^2$.

Introduction:
(For use on overhead)

Given a sheet of paper, how many times can I fold it? Would you say 6 or 7 times! Notice the initial rectangular shape decreases as the number of folds increases. Therefore, the number of rectangles increases as the number of folds increases.
How many rectangles are created?

The first fold of the paper creates two rectangles. The second fold creates four. If the paper is folded a third time, eight rectangles are formed.

Question: How many rectangles are formed on the fourth fold?

Answer: Sixteen. Notice that for each fold the number of rectangles double from the previous number.

If a table is created, the following pattern develops.

Folds	0	1	2	3	4	5	6	7	8	9	10
Rectangles	1	2	4	8	16	32	64	128	256	512	1024

Could the data be written in a different form? What about exponents?

If the form a^x is used, a represents the base and x the exponent. Then, in this situation, two would be our base and the number of folds represented by x. The new table of values would be

Folds	0	1	2	3	4	5	6	7	8	9	10
Rectangles	2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}

Question: How many rectangles are created after 30 folds?

Answer: 2^{30} or 1,073,741,824 rectangles.

This could be written as a function: $Y = 2^x$

Using a calculator, it is easy to calculate the number of rectangles as a number is substituted for x.

A TI-83 (Plus) can be used to show how this function is represented graphically.

Procedure for Producing Graphs:

Before hand make sure that:

a. Press [MODE], and everything is highlighted to the left.
b. Press [2nd], [FORMAT], and everything is highlighted to the left.
c. Press [Y=], make sure there aren't any equations in your y= and make sure none of your plots are highlighted (if it is highlighted the plot is on).
d. Press [ZOOM], select 6 for the standard Cartesian Plane.

Date: _____

Names: _____

Group Activity Sheet

You are at an interview for a job. While there, you are asked which salary you would prefer. Salary A would pay $500 the first day and give you a $100 daily increase. Salary B would pay 1 cent for the first day and every successive day you will be paid double the amount.

Questions:

1. Create a table of values that would show patterns for salary A and salary B.

# of Days	1	2	3	4	5	10	15	....	19
Pay A									
Pay B									

2. Determine the function for each salary.
   Payment A:

   Payment B:

3. By using the equations, sketch the graphs of the two functions. Tell where the lines intersect and explain what it means.

4 Would you rather be given salary A or salary B? Explain.

Group Activity Sheet Key

You are at an interview for a job. While there, you are asked which salary you would prefer. Salary A would pay $500 the first day and give you a $100 daily increase. Salary B would pay 1 cent for the first day and every successive day you will be paid double the amount.

Questions:

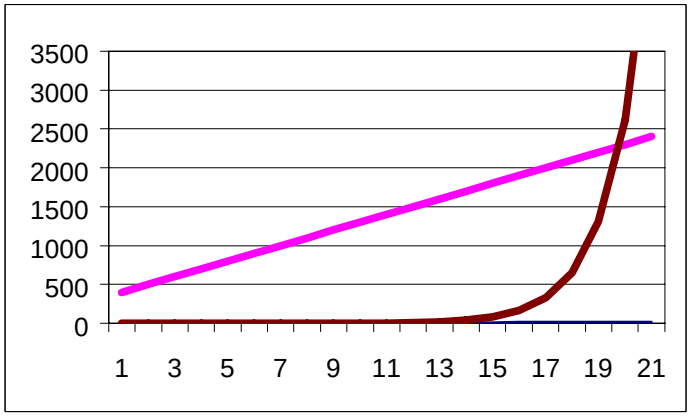
1. Create a table of values that would show patterns for salary A and salary B.

# of Days	1	2	3	4	5	10	15	....	19
Pay A	500	600	700	800	900	1400	1900		2300
Pay B	0.01	0.02	0.04	0.08	0.16	5.12	163.84		2621.44

2. Determine the function for each salary.

   Salary A: $y = 100(x-1) + 500$
   Salary B: $y = .01^{(2x-1)}$

3. By using the equations, sketch the graphs of the two functions. Tell where the lines intersect an explain what it means.



The two lines intersect. This means about the 19th day both salary A and salary B will equal .

4. Would you rather be given salary A or salary B? Explain.
   It all depends . If you are on the job more than 18 days you would start to earn more money per day, if you use salary B. (However don't forget the total money you have earned in the past. At day 19, salary A yields $26,600 versus salary B whose total is $5,242.80.

Day-2
Warm Up

Evaluate the following:
a) 2^5 b) 5^2 c) $3^2 * 4^3$ d) $x^3 * x^4$ e) $(x^3)^4$ f) $(2x^3)^4$ g) $(3x^2)^5$

Key:
a) 32 b) 25 c) 576 d) x^7 e) x^{12} f) $16x^{12}$ g) $243x^{10}$

Introduction:
John has $100 which he invests at 8% interest. How much money will he have in a) 3 years? b) 5 years? c) 10 years? d) 20 years? e) 50 years f) 80 years? When will he have g) $1,000 ? h) $10,000?

Introduction Key:
a) $116.64 b) $146.93 c) $215.89 d) $466.10 e) $4,690.20 f) 47,195.48
g) 29.1919 yrs. h) 59.838 yrs.

Note: See if any students are able to work through the last two problems from the introduction. Show the students how to solve them graphically on the TI-83 (Plus). Follow these steps:

1. Press [Y=] then type 100(1.08)^ x in Y1.
2. Place 1000 in Y2.
3. Press [WINDOW]. Set Xmin = 0, Xmax = 40, Ymin = 0, and Ymax = 1200.
4. Press [GRAPH].
5. Press [2^{nd}] [CALC] , select 5: intersect .
6. Move the cursors near the point of intersection and press [ENTER]. Do the same for Y2. Press [ENTER] again . Press [ENTER] a third time and the point of intersection will appear.

At a 6% rate of return how long it will take $10,000 to a) double, b) quadruple, and c)increase ten times the original value.

Key: a) 11.896 yrs. b) 23.791 yrs. c) 39.517 yrs.

Day-2
Identifying Functions

NOTE: This is a teacher directed, group learning activity, designed to demonstrate how to identify functions by looking at the formula, graph, or numerical pattern. The teacher should feel free to add additional examples and modify this section as he/she feels necessary to meet their individual class needs. (It is assumed that linear and quadratic functions have been taught.)

I. Look at the functions $y_1 = 3x + 2$, $y_2 = x^2 + 2x + 3$, and $y_3 = 2^x$.

1. What type of function does each represent?
2. Graph each function on the graphic calculator. Compare and contrast each graph.
3. What characteristics do you notice about each graph?

II. Using the table function on the graphic calculator, make a table for each function, using values of x between –3 and 3 inclusive. Make the change of x by 1. Have the student's copy the table on their paper.

x	-3	-2	-1	0	1	2	3
$y = 3x + 2$	-7	-4	-1	2	5	8	11
$y = x^2 + 2x + 3$	6	3	2	3	6	11	18
$y = 2^x$	.125	.25	.5	1	2	4	8

a) Look at the change of the linear function. What do you notice? The change in the value of y is 3 everytime the value of x increases by 1. (Students should have learned this in the linear functions unit, but may need review.)

b) Look at the change of the quadratic function. Students should recall from the quadratic unit that this function's characteristic is produced by the second difference.

x	-3	-2	-1	0	1	2	3
f(x)	6	3	2	3	6	11	18
1^{st} difference	-3	-1	1	3	5	7	
2^{nd} difference		2	2	2	2	2	

c) Look at the change of the exponential function. See if the students can discover the pattern. You may need to guide students into looking at the division of values of f(x). Use f(3)/f(2) as a starting point. Work the students towards the conclusion that $a^x/a^{x-1} = a$. This test will identify an exponential function.

x	-3	-2	-1	0	1	2	3
f(x)	.125	.25	.5	1	2	4	8
f(x)/f(x-1)		2	2	2	2	2	2

At the conclusion of this lesson, the student should be able to identify the type of function by equation, table, or graph. Several working problems may be introduced by the teacher to reinforce this section. The students will be practicing these skills for homework.

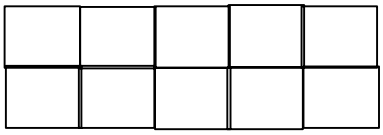
Activity Sheet

1. Fill in the chart below by giving the characteristic of each type of function.

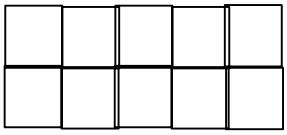
Function Type	Numerical	Equation	Graphic
Linear			
Quadratic			
Exponential			

2. Identify the following types of functions. Verify your reason.

a)



b)



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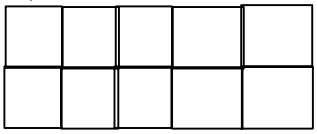
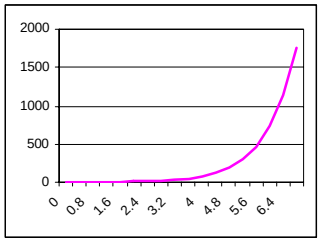


d) $y = 5^x$

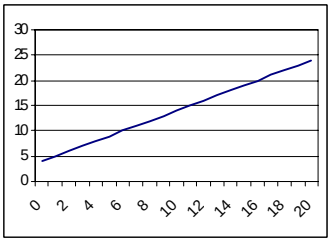
e) $y = (1/2)x - 3$

f) $y = 3x^2 + 7x - 2$

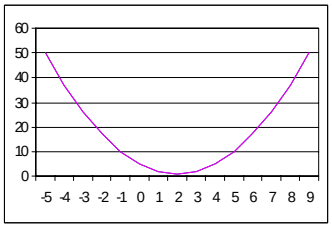
g)



h)



i)



Activity - Key

1.

Function Type	Numerical	Equation	Graphic
Linear	constant rate of change $\Delta y/\Delta x$	$y = mx + b$	line
Quadratic	second difference	$y = ax^2 + bx + c$	parabola
Exponential	a^x / a^{x-1} = constant	$y = a^x$	exponential curve

2. a) Linear b) Quadratic c) Exponential d) Exponential e) Linear f) Quadratic
g) Exponential h) Linear I) Quadratic

Homework Assignment – Day 2

1. Bobby wants to compare the growth of a stock fund yielding 12% (hopefully) with a bank account yielding 4%. Start with $100.00 and grow it at 5 year intervals for 30 years.

2. Make up problems representing each type of function using each representation. You will exchange these tomorrow with members of your team, so make them challenging.

Home Assignment – Key

1.

Time	0	5	10	15	20	25	30
12%	500	881.17	1552.92	2736.78	4823.14	8500.03	14980.96
4%	500	608.33	740.12	900.47	1095.56	1332.92	1621.7

2. Answers will vary.

Day 3
Warm Up

1. Graph and sketch the following exponential functions.
   a) $y = 3^x$ b) $y = 5 * 3^x$ c) $y = 1000 * 3^x$

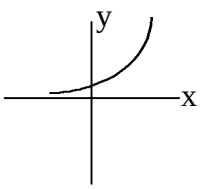
2. Each graph is increasing. What could be done to the equation to make the graph decrease?

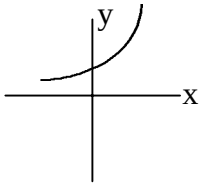
3. In the exponential function $y = a^x$, can the value of a be 0 or 1?

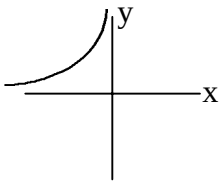
Day 3
Warm Up - Key

1. Graph and sketch the following exponential functions.
   a) $y = 3^x$ b) $y = 5 * 3^x$ c) $y = 1000 * 3^x$

a) b) c)

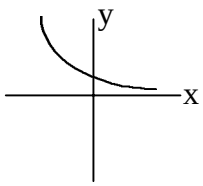


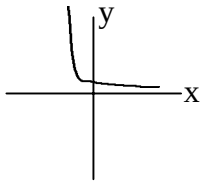




2. Each graph is increasing. What could be done to the equation to make the graph decrease?

   Change the value of a to some number between 0 and 1. $0 < a < 1$. For example:





3. In the exponential function $y = a^x$, can the value of a, be 0 or 1?

   No, the value of a can not equal 0 (for $x > 0$) since $y = 0^x$ will result in a linear function on the x-axis and $y = 1^x$ will result in a linear function.

Introduction
(Transparency)

You won a brand new car from a raffle. The car is worth $35,000. Unfortunately, you can not drive it, because you do not have a license. You wait 5 years to get your license, by then the car is not new. So you want to sell it. Well cars decrease in value 20% each year. How much would the car be worth the first day of…
a) 1 yr. b) 2 yrs. c) 3 yrs. d) 4 yrs. e) 5 yrs. f) 10 yrs. g) 15 yrs.

Enter your data in L1 (Year) and L2 (Value), create a graph to see what type of function Is being shown.

TI-83 (Plus) procedure:

1. Press [STAT], enter 1: Edit.
2. Enter data. Be sure each list has the same amount of data.
3. Press [2nd] [STAT PLOT]. Enter 1: Plot 1.
4. Turn on the plot, Type of graph: Line graph, Xlist: L1, Ylist: L2, and the Mark should be the square.
4. Press [ZOOM] , enter 9: ZoomStat.
5. Press [GRAPH].
6. Press [2nd] [STATPLOT]. Enter 1: Plot 1. Change your Type of graph: Scatter Plot.
7. Press [Graph].
8. To determine the equation, press [STAT], scroll over to CALC, enter 0: ExpReg. Press [ENTER]. (Your screen should show ExpReg).
9. Enter [2nd] L1, comma, [2nd] L2, comma, [VARS], scroll over to Y-VARS, Enter 1: Function, then enter 1: Y1. Press [ENTER], [ENTER].
10. Press [Y=] to view the function.

Group Activity Sheet:
Day 3

1. Complete the table of values. (Hint use percent decrease)

1	2	3	4	5	6	7	8
227.5	147.87	96.12	62.47				

2. Make a scatter plot of the values in L1 and L2, then sketch.

3. Using exponential regression determine the function. Identify the initial amount, decay factor, and rate of decrease.

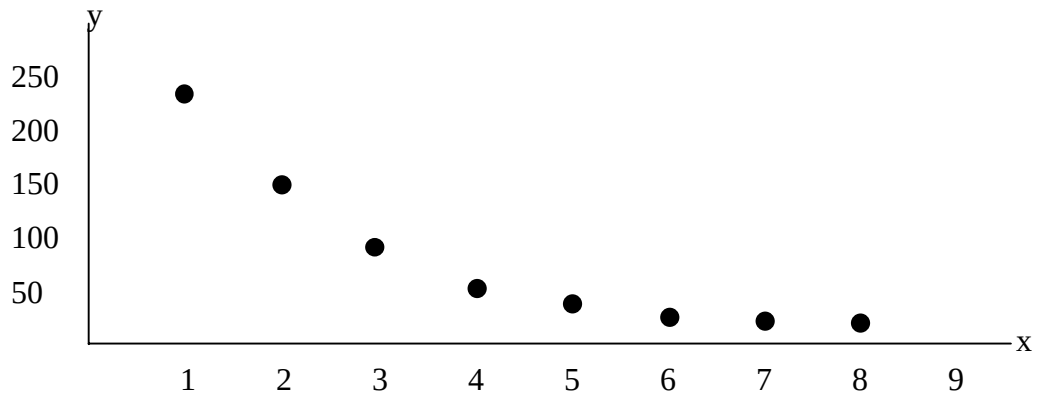
4. Create a situation for which your equation will be true.

Group Activity Sheet - Key
Day 3

1. Complete the table of values. (Hint use percent decrease)

1	2	3	4	5	6	7	8
227.5	147.87	96.12	62.47	40.61	26.39	17.15	11.14

4. Make a scatter plot of the values in L1 and L2 and sketch.



5. Using exponential regression determine the function. Identify the initial amount, decay factor, and rate of decrease.
Function: $y = 350 * (.65)^x$. Initial amount = 350. Decay factor = .65
Percent of decrease = 35%.

4. Create a situation for which your equation will be true.
Answers will vary.

Homework Assignment – Day 3

Your job as an exterminator is to get rid of the vermin in a given area.
If they are no longer able to reproduce, how many treatments will it take for you to get rid of 90% of the population? At present there are estimated 3,500,000 and each treatment will eliminate 30%. Create a table of values.

Algebra I - Assessment

Part I - Selected Response

1. Given the table of values, which of the following functions could be used to express the relationship between x and f (x)?

x	0	1	2	3
f(x)	2	6	18	54

a) $f(x) = ax^2 + bx + c$ b) $f(x) = bx + a$

c) $f(x) = ab^x$ d) $f(x) = a \bullet |x| + b$

2. Given the table of values, which of the following functions could be used to express the relationship between x and y?

x	0	1	2	3
f(x)	2	-1	0	5

a) $y = ax^2 + bx + c$ b) $y = bx + a$

c) $y = ab^x$ d) $y = a \bullet |x| + b$

3. Which of the following functions will represent $500 placed into a mutual fund yielding 10% per year for 4 years.

a) $A = 500 (.10)^4$ b) $A = 500 (1.1)^4$

c) $A = 500 (4) (.10)$ d) $A = 500 (1.04)^{10}$

4. Given the table below:

x	0	1
g(x)	5	15

If the function is exponential find g (2).

a) 25 b) 30 c) 45 d) 225

Part II - Student Produced Response

5. What is the value of $f (x) = ab^x$, when $a = 2$, $b = 5$, and $x = 3$?

6. What is the value of the y - intercept of $y = 5a^x + 2$?

7. John pays 10% of his outstanding balance on his credit card per year. If his beginning balance was $1,000, how much does he owe at the end of 3 years?

Part III - Brief Constructed Response (3 points)

8. The projected population of Delroysville is given by the formula $p (t) = 1500 (1.08)^t$. You have been selected by the city council to help them plan for future growth.

   a) Explain what the formula $p (t) = 1500 (1.08)^t$ means to the city council members.

   b) Make a table of values to show the population growth for the next 5 years. Round each answer to the nearest whole number.

   c) It is now December 1, 2001, the city hopes to have a population of 3,000 people by 2010. When will the city reach their goal? Show all work and give specific dates.

   d) A councilperson objects to your growth rate and feels 10% growth is more realistic. Write a function showing this new projected rate and show how long it will take the population to double.

Exponent Quiz
Answer Sheet

Name _____

Date _____ Period _____

1) _____

2) _____

3) _____

4) _____

5) _____

6) _____

7) _____

8) a. _____

b.

Year	0	1	2	3	4	5
Population						

c. _____

d. _____

Assessment - Key

1. C

2. A

3. C

4. C

5. 250

6. 7

7. 729

8. a) 1500 is the initial population on December 1, 2001. 1.08 is the rate of growth, which is 8% per year and t is the number of years.

   b)

Year	0	1	2	3	4	5
Population	1500	1620	1750	1890	2041	2204

   c) Use a graphing calculator to find the intersection of p(t) = 3000. This occurs at 9.0064683. The city will meet their projection.

   d) The new function is $p(t) = 1500\ (1.1)^t$. It will take 7.2725409 years to reach 3000 which will happen approximately March 10, 2009 at 11: 27.